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CRACK POINT STRESS SINGULARITIES AT A BI-MATERIAL INTERFACE

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The mathematical procedure for analyzing stress singularities in infinite wedges has been developed in References (1) and (2) and has been successfully applied to the analysis of stress distribution in the vicinity of a tip of a crack[3] [4]*. As a continuation of this study, the results presented in this note relate to the symmetrical stress field about a crack point perpendicular to a bi-material interface, and hence complement earlier results[4] wherein the crack lay along the interface.

The present analysis pertains to a crack in a medium M_1 which terminates perpendicularly at the interface between M_1 and M_2 as shown in Figure 1. Introducing product solutions of the bi-harmonic equation of the type $X_i(r, \psi) = r^{\lambda+1} F_i(\psi : \lambda)$, i=1, 2, and following the notation and procedure of Reference 4, the normal and tangential stresses and displacements are matched on the interface and the free edge stress condition is imposed on the faces of the crack. Specifically one finds[5] in the respective regions, sets of eigenfunctions of the form

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* Superscripts refer to references at the end of note.

$$\frac{X_1(r,\psi)}{d_2 r^{\lambda+1}} =$$

$$\left\{\left[\frac{(3\lambda-1)\alpha+(1-2\lambda)\beta+\lambda}{(\lambda+1)(\alpha+1)}\right]\left[(\lambda-1)\alpha-1\right]-\frac{\lambda-1}{\lambda+1}\left[(\lambda-1)\alpha+\beta-1\right]\right\}\cos\lambda\frac{\pi}{2}\sin(\lambda+1)\psi$$

$$+\left\{\left[\frac{(3\lambda-1)\alpha+(1-2\lambda)\beta+\lambda}{(\lambda+1)(\alpha+1)}\right]\left[-(\lambda+1)\alpha-1\right]+\left[(\lambda+1)\alpha-\beta+1\right]\right\}\sin\lambda\frac{\pi}{2}\cos(\lambda+1)\psi$$

$$+\left\{\left[\frac{(3\lambda-1)\alpha+(1-2\lambda)\beta+\lambda}{(\lambda+1)(\alpha+1)}\right]\left[-(\lambda+1)\alpha\right]+\left[(\lambda-1)\alpha+\beta\right]\right\}\cos\lambda\frac{\pi}{2}\sin(\lambda-1)\psi$$

$$+\left\{\left[\frac{(3\lambda-1)\alpha+(1-2\lambda)\beta+\lambda}{(\lambda+1)(\alpha+1)}\right]\left[(\lambda+1)\alpha\right]+\left[-(\lambda+1)\alpha+\beta\right]\right\}\sin\lambda\frac{\pi}{2}\cos(\lambda-1)\psi \qquad (1)$$

$$\frac{X_2(r,\psi)}{d_2 r^{\lambda+1}} =$$

$$\left[\frac{(3\lambda-1)\alpha+(1-2\lambda)\beta+\lambda}{(\lambda+1)(\alpha+1)}\right]\cos(\lambda+1)\psi+\cos(\lambda-1)\psi \qquad (2)$$

where d_2 is a constant which depends on the type of the external loading, and the eigen parameters, λ, which may be real or complex are found from the characteristic equation derived from the homogeneous boundary conditions,

$$\left\{\lambda^2(-4\alpha^2+4\alpha\beta)+2\alpha^2-2\alpha\beta+2\alpha-\beta+1\right.$$

$$\left.+(-2\alpha^2+2\alpha\beta-2\alpha+2\beta)\cos\lambda\pi\right\}\sin\lambda\pi = 0 \qquad (3)$$

with the material property parameters defined as

$$\alpha = \frac{k-1}{4(1-\sigma_1)}, \quad \beta = \frac{1-\sigma_2}{1-\sigma_1} k, \quad k = \frac{\mu_1}{\mu_2} \tag{4}$$

and assuming plane stress conditions

$$\sigma_i = \frac{\nu_i}{1+\nu_i}$$

RESULTS AND DISCUSSION

Each solution of (3) defines an eigenfunction, and an infinite set of $X_n(r, \psi)$ may be generated. For the present purposes however, we are concerned only with those values of λ which may lead to singularities in the stress at the crack point. It can easily be shown that this condition corresponds to a requirement that the real part of λ lie between 0 and 1. For this particular problem it develops that the minimum value of λ satisfying this condition is real and in Figure 1 its value is given as a function of the ratio of the shear moduli of the two regions, k, assuming for computational simplicity that the Poisson's ratio, ν, of the two materials is the same and is equal to 0.3.

It is interesting to note that as material M_1 becomes harder with respect to M_2, that is M_1 has larger elastic modulus than M_2, the strength of the singularity, which is equal to $(\lambda - 1)$, increases. In fact in the limit as $k \to \infty$ it can be shown that the strongest singularity, $\lambda \to 0$, is produced.

In actual physical situations the high stress concentration at the tip of the crack leads invariably to plastic deformation. To obtain an idea of the area of plastic deformation, consider an elastic estimate based on a von Mises criterion, i.e. the locus of constant distortional strain energy, $W_d(r, \psi) =$ constant. The distortional energy distribution corresponding to the lowest eigenfunction is shown in Figure 2. In calculating these curves, plane stress conditions ($\sigma_3 = 0$) have been assumed and results are presented for three different values of the elastic moduli ratio $k = 1/20$, 1, 20, where $k = 1$ corresponds to the homogeneous situation where $M_1 = M_2$. It can be seen that for $k = 1/20$ the maximum distortional energy lies along rays at approximately $\psi = \pm 70^\circ$, and hence is essentially unchanged from the homogeneous case, while for $k = 20$, i.e. the crack entering a softer medium, the maximum occurs at the interface.

In Reference (5) the principal stress distribution has also been calculated and the results of this calculation can be summarized as follows. In the homogeneous case, the maximum of the principal stress was found to occur ahead of the crack at $\pm 60^\circ$ to the direction of prolongation. This same tendency also prevails when the crack is in the softer of the two materials, even to the angular position. On the other hand when the crack is proceeding from the hard into the softer material, the maximum stress occurs along the interface and is nearly an order of magnitude larger than the largest principal stress ahead of the crack. Also, it was found in the homogeneous case that the principal stresses near the crack point along $\psi = 0$

were equal, thus leading to a state of "two-dimensional" hydrostatic tension and consequently less yielding. For the non-homogeneous situation, the ratio of the stresses σ_r and σ_ψ differ from unity, actually being

$$\frac{\sigma_r(r,0)}{\sigma_\psi(r,0)} = \frac{4 + 2\left(\frac{\mu_1}{\mu_2} - 1\right)(1+\nu_1) + (2\lambda - 1)\left[\frac{E_1}{E_2}(1-\nu_2) - (1-\nu_1)\right]}{4 - (2\lambda - 1)\left[\frac{E_1}{E_2}(1-\nu_2) - (1-\nu_1)\right]} \tag{5}$$

which is seen to approach unity when $M_1 \longrightarrow M_2$. Presumably therefore, a larger plastic region at the crack tip might be expected.

The behavior of the distortional energy seems to agree with physical experience, with the situation k = 20 leading toward a sort of peeling away of the softer material perpendicular to the crack direction. On the other hand it should be observed[5] that there is a small range of rigidity ratios for which the maximum stress at the interface ($\psi = \pi/2$) is still less than the absolute maximum at $\psi = 60^\circ$.

In conclusion two brief remarks are in order: (1) the stress distributions have been arbitrarily limited to symmetrical loadings, and (2) the lowest eigen value for this example is real and thus gives strictly a monotonic decay of stress, $\sigma \sim r^{-a}$. This latter behavior is characteristically different from the associated case[4] when the crack lies along the interface. Here the lowest eigen value is complex, and leads to a damped, trigonometric variation, e.g. $\sigma \sim r^{-a} \cos(b \log r)$. It seems reasonable therefore to assume that at some intermediate inclination of the crack to the interface, the characteristic behavior will switch from one to the other, as controlled by the relative value of the smallest positive real part of the complex eigen value to the real root.

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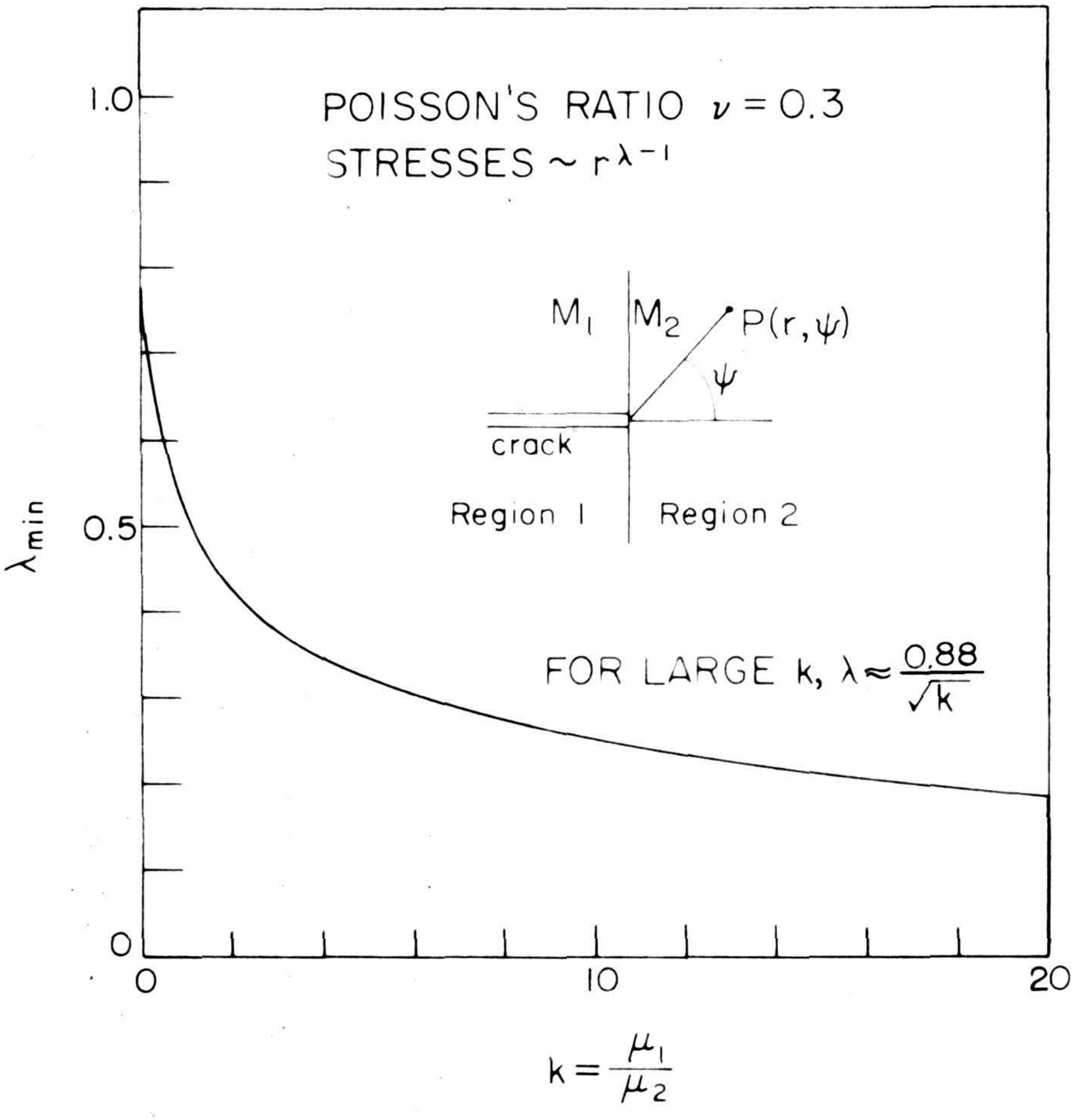

FIG. 1 MINIMUM EIGENVALUE

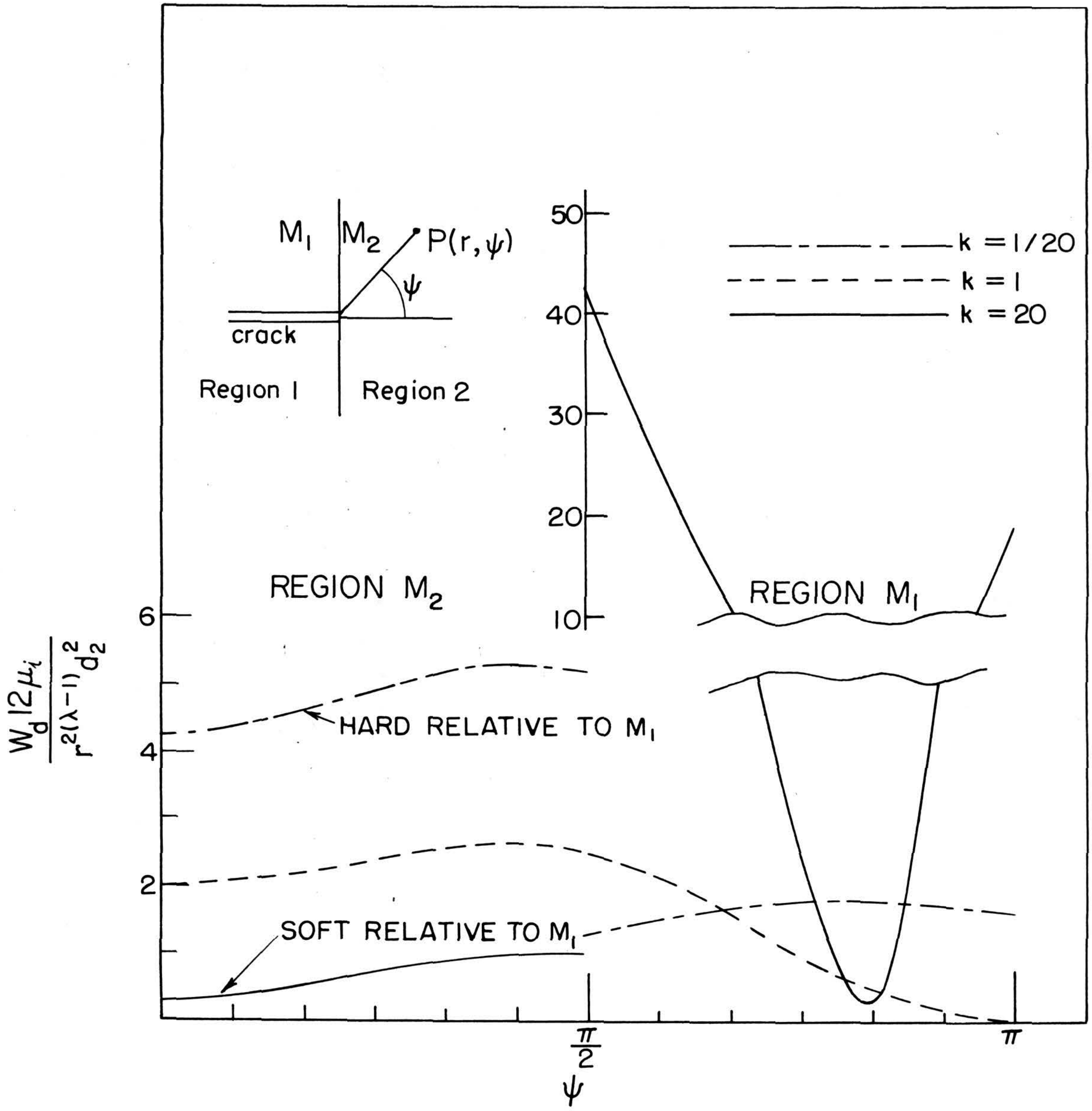

FIG. 2 DISTORTION STRAIN ENERGY DISTRIBUTION